

# Akhil on reduction, gap + slice theorems

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Consider  $MU^{((G))} = \bigvee_{C_2}^{G_1} MU_{\mathbb{R}}$  for  $G = C_8$

$\exists D \in \pi_{19P_8}^{G_1} MU^{((G))}$  and let  $S_0 = MU^{((G))}[D^{-1}]$ ,

$$Q = S_0^{hC_8} = S_0^{C_8} \quad (\text{equality due to thm})$$

Gap Theorem  $\pi_i S_0 = 0$  for  $-4 < i < 0$

$S_0 = \text{hocolim} (MU^{((G))} \xrightarrow{D} \Sigma^{19P_8} MU^{((G))} \rightarrow \dots)$

In fact  $\pi_i^{C_8} \Sigma^n P_8 MU^{((G))} = 0$  for  $-4 < i < 0$  for all  $n$ .

This is true for a class of  $G$ -spectra. It is proved using the slice.

For now let  $G = C_{2^n}$

Def A slice is pure and cellular if it is a wedge of summands

$\tilde{S}(m, K) \wedge H \geq$  where

Def A  $G$ -spectrum  $X$  is pure and isotropic if all of its slices have this form with  $K \neq \ell$ .

Slice Thm  $MU^{((G))}$  is pure + isotropic

General Gap Theorem If a  $G$ -spectrum (for  $G \neq e$ ) is pure and isotropic, then  $\pi_i^G X = 0$  for  $-4 < i < 0$ .

If  $X$  is pure + isotropic, so is  $\sum m P_{n+1} X$ .

This implies the Gap for  $S_0$

Example  $K_{\mathbb{R}}$  as a  $\mathbb{G}$  spectrum, slices are  $H\mathbb{Z}_1 S^{mp_2}$   
 Gap theorem holds.

Let  $X$  be pure + isotropic.  $X = \lim P^n X$  with  
 $P^n X = \bigvee H\mathbb{Z} \wedge \tilde{S}(m, k)$  with  $m|k| = 0, k \neq e$

Suffices to show  $\pi_i^{G_1} P_n X = 0$  for  $-4 < i < 0$

Enough to show  $0 = \pi_i^{G_1} H\mathbb{Z}_1 / (G_1 \wedge S^{mp_k}) = \pi_i^k H\mathbb{Z}_1 S^{mp_k}$

Cell Lemma For  $G_1 \neq e$ ,  $\pi_i H\mathbb{Z}_1 S^{mp_G} = 0$  for  $i \in (-4, 0)$

PF  $[S^i, H\mathbb{Z}_1 S^{mp_G}]^{G_1} = [\Sigma^i S^{mp_G}, H\mathbb{Z}]^{G_1}$  and  
 $[\Sigma^i X, H\mathbb{Z}]^{G_1} \cong \tilde{H}^i(X/G_1; \mathbb{Z})$  *This does not work with homology*

Consider  $[\Sigma^i S^{mp_G}; H\mathbb{Z}]^{G_1} \cong [S^0, H\mathbb{Z}_1 S^{i+1} S^{mp_G}]$

vanishes for  $m \geq 0$  and  $i$  as above

Let  $m = -l$  for  $l > 0$ .

$[\Sigma^i S^{lp_G}; H\mathbb{Z}] \cong \tilde{H}^{-i}(S^{lp_G}/G_1; \mathbb{Z})$  for  $i = -1, -2$ .

For  $l \geq 2$ ,  $S^{lp_G} = \sum^2 (-)$  so we need only consider  $l=1, i=-2$ . Need  $H^2(S^{lp_G}/G_1; \mathbb{Z}) = 0$

$P_G = 1 + \tilde{P}_G$        $H^1(S^{lp_G}/G_1; \mathbb{Z})$

There is the Euler sequence

$$S(\tilde{P}_G)_+ \rightarrow S^0 \rightarrow S^{\tilde{P}_G}$$

$$\text{and } \tilde{s}^{\rho_\alpha}/\zeta = \sum (\zeta(\tilde{\rho}_\alpha)/\zeta)$$

To understand slice tower of  $MU^{((G))}$ , build  $P^n MU^{((G))}$   
by "algebraic methods"

Ex  $R = \text{ring spectrum with } \pi_* R = \mathbb{Z}[x], x \in \pi_2 R$

$$R/x := \text{cofiber } (\Sigma^2 R \xrightarrow{x} R) = P^0 R$$

$$R/x^n := \dots (\Sigma^{2n} R \xrightarrow{\quad} R) = P^{2n} R$$

Ex  $R = E_2 \text{ ring with } \pi_2 \mathbb{Z}[x, y], |x|=2, |y|=4$ .

Let  $A_1 = S^0[S^2] = \bigvee_{i \geq 0} S^{2i} \exists \text{ map } A_1 \rightarrow R \text{ of algebras}$   
 $S^2 \mapsto 1 \text{ algebra}$

$$A_2 = S^0[S^4] = \bigvee_{i \geq 0} S^{4i} \quad A_2 \rightarrow R \quad " \quad "$$

$\begin{matrix} A_1 \wedge A_2 \rightarrow R \\ \parallel \end{matrix} \quad R \rightarrow R \quad A = A_1 \wedge A_2 \text{ is an associative ring}$

$\bigvee_{i,j \geq 0} S^{2i+4j} \text{ Let } I_{2n} = \bigvee_{\substack{i,j \\ 2i+4j=2n}} S^{2i+4j} = A\text{-bimodule}$

We have a tower  $\{A/I_{2n}\}$ .

$$I_{2n}/I_{2n+2} = \bigvee_{\substack{i,j \\ 2i+4j=2n}} S^{2n}$$

Then  $R_A S^0 = \# \mathbb{Z} \text{ and } R_A(A/I_{2n}) \cong P^{2n-2} R$

Thm  $\pi_*^U MU^{((G))} = \mathbb{Z}[G \cdot m_1, G \cdot m_2, \dots]$  where

$$G \cdot m_i = \{m_i, \gamma m_i, \gamma^3 m_i, \dots \gamma^{\lfloor G/2 \rfloor - 1} m_i\}$$

Each  $m_i$  refines to  $S^i P_2 \xrightarrow{\pi_2} \text{res}_{C_2}^G MU^{((G))}$

For each  $i$ ,  $S^0[S^{ip_2}] = \bigvee_{j \geq 0} S^{ijp_2}$

$S^0[G \cdot \bar{\gamma}_1] = N_{C_2}^{G_1} S^0[S^{ip_2}]$  = indexed smash product

For each  $i$  we have a map  $S^0[G \cdot \bar{\gamma}_1] \rightarrow MU^{((G))}$   
 by taking norm of  $S^0[\bar{\gamma}_1] \rightarrow MU_{\mathbb{R}}$ . Smashing  
 these for all  $i > 0$  we get

$A^0 := S^0[G \cdot \bar{\gamma}_1 : i > 0] = \bigwedge_{i > 0} S^0[G \cdot \bar{\gamma}_1] \rightarrow MU^{((G))}$

This is a refinement of homotopy.

Reduction Thm  $MU^{((G))} \xrightarrow[A]{} S^0 \cong H\mathbb{Z}$

(due to Hu-Kriz for  $G = C_2$ , HHR for  $G = C_2$ ,  
 motivic analog due to Hopkins-Morel, Hoyois.)

Proof is by induction on  $|G|$  and is not easy.

Some details below

Derivation of Slice Theorem from Reduction Thm

$A = S^0[\{G \cdot \bar{\gamma}_1 : i > 0\}] = \bigvee S^{P_f}$

1)  $J = \coprod_{i > 0} G/C_2$ ,  $f: J \rightarrow \mathbb{Z}_{\geq 0}$   
 fin-supported

2)  $K_f$  = stabilizing gp of  $f$

$P_f$  = multiplicity of reg rep of  $K_f$  of dimension

$$2 \sum_{j \in J} i f(j)$$

Let  $I_{2n} \subseteq A$  = wedge of all spheres of dim  $\geq 2n$ .

Then  $I_{2n}/I_{2n+2}$  = wedge of isotropic slice cells of  $\dim^{2n}$  <sup>adapted on thru</sup>  
dimension  $\geq n$  <sup>augmentation</sup>

Consider the tower  $MU^{((G))} \wedge_{A/I_{2n}} (A/I_{2n}) = P^{2n-2} MU^{((G))}$

Pf: Associated graded

$$MU^{((G))} \wedge_A (I_{2n}/I_{2n+2}) \simeq (MU^{((G))} \wedge_A S^0) \wedge I_{2n}/I_{2n+2} \\ \simeq H\mathbb{Z} \wedge (\text{slice cells})$$

Pf of Reduction Thm. For  $|G| \leq 2$  it is classical.

$$\text{Red}(C_{2^n}) \rightarrow \text{Slice}(C_{2^n})$$

$$\text{Red}(C_{2^{n+1}}) \xrightarrow{\quad} \text{Slice}(C_{2^{n+1}})$$

Suppose we know it for proper subgp

Need to show

$$\underline{\Phi}^G (MU^{((G))} \wedge_A S^0) \cong \underline{\Phi}^G H\mathbb{Z}$$

$$MU \wedge_A S^0 \\ S^0[h_2, h_4, h_5, \dots]$$

$\mathbb{Z}/2[\ell]$  with  $|\ell|=2$ .

Isotropy separation reduces to this case.